— Exercises —

1. **Harmonic functions.** The *Laplacian* of a real-valued $\mathcal{C}^2$ function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is defined as $\Delta f(p) := \sum_{k=1}^n D_{kk} f(p)$. The function f is called *harmonic* when $\forall p, \Delta f(p) = 0$. For the following functions f, compute Δf and determine whether f is harmonic or not.

   (a) $f(x_1, x_2) = e^{x_1} \sin x_2$ on $U = \mathbb{R}^2$,

   (b)* $f(x_1, \ldots, x_n) = (x_1^2 + \cdots + x_n^2)^\alpha$ on $U = \mathbb{R}^n - \{0\}$, for some real constant α.

2. **Examples for the MVT.** For every point $x = (x_1, x_2)$ in $\mathbb{R}^2$ which does not belong to the line $D : x_1 + x_2 = 0$, we put $f(x) = \frac{x_1 x_2}{x_1 + x_2}$.

   (a) Explain briefly why f is differentiable on $\mathbb{R}^2 \setminus D$ and compute $\nabla f(x)$ at any point x.

   (b) If $x = (1, 1)$ and $y = (2, 4)$, find all the points z on the segment $[x, y]$ such that

   $$< \nabla f(z), y - x >= f(y) - f(x)$$

   (c) Same question with $x = (1, 1)$ and $y = (2, 2)$. And again the same question with $x = (1, 1)$ and $y = (-2, 1)$.

3. **A partial differential equation.** Using the change of variables $\begin{cases} u = x + y \\ v = 2x + 3y \end{cases}$, find all $\mathcal{C}^2$ functions defined on $\mathbb{R}^2$ such that $3\frac{\partial f}{\partial x} - 2\frac{\partial f}{\partial y} = 0$.

— Problems —

4. **Radial harmonic functions.** Let g be a $\mathcal{C}^2$ function on $\mathbb{R}_{>0}$. We define a function on $\mathbb{R}^2 \setminus \{0\}$ by $f(x) = g(||x||^2)$ where $||x||$ is the Euclidean norm of x.

   (a) Show that f is $\mathcal{C}^2$ and compute its Laplacian.

   (b) We assume now that f is harmonic. Prove that $h = g'$ satisfies the differential equation $h(r) + rh'(r) = 0 \quad (*)$.

   (c) Solve $(*)$ and find the form of f.

5. **Local vs. global diffeomorphism.** Let f be the function defined by

   $$\begin{aligned} f : \mathbb{R}^2 - \{(0,0)\} &\to \mathbb{R}^2 - \{(0,0)\} \\ (x, y) &\mapsto (x^2 - y^2, 2xy). \end{aligned}$$

   We will show that f is a *local diffeomorphism*, i.e. $\forall p$ in the domain, $\exists N$ a neighborhood of p such that $f_{|N}$ is a differentiable bijection onto its image, whose inverse is also differentiable.

   (a) Show that $||f(x, y)|| = ||(x, y)||^2$ and that $f(x, y) = f(x', y')$ if and only if we have $(x, y) = (x', y')$ or $(x, y) = -(x', y')$.

   (b) Consider the upper half plane $H = \{(x, y) | y > 0\}$. Check that f in injective on H and that $I = f(H) = \mathbb{R}^2 \setminus \{(u, v) | u \geq 0, v = 0\}$. *Hint: polar coordinates.*

   (c) Show that g is differentiable. *Hint: find $g : I \to H$ s.t. $f \circ g$ is the identity.*

   (d) Denote by R_α the rotation of the plane of angle α about the origin. Check that $\forall \alpha, f \circ R_\alpha = R_{2\alpha} \circ f$ and show that f is a local diffeomorphism. Is f injective on $\mathbb{R}^2 \setminus \{0\}$?